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Cardinal Lacunary Interpolation by g-Splines. I. The Characteristic Polynomials

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1. INTRODUCTION

Let n, r be positive integers such that $r \leqslant [n/2]$ and let $\mathscr{S}_{n,r}$ denote the class of Cardinal spline functions of degree n with integer knots of multiplicity r. More precisely, $S(x) \in \mathscr{S}_{n,r}$ if

$$S(x) \in C^{n-r}(-\infty, \infty) \tag{1.1}$$

and

$$S(x) \text{ is equal to a polynomial of degree } n \text{ in each of the intervals } [\nu, \nu+1], \quad (\nu = 0, \pm 1, \pm 2,\ldots). \tag{1.2}$$

The Cardinal Hermite interpolation problem (C.H.I.P.) is posed as follows: Given r bi-infinite sequences of numbers

$$y = (y_\nu), \qquad y' = (y_\nu'),\ldots, y^{(r-1)} = (y_\nu^{(r-1)}) \tag{1.3}$$

we wish to find an $S(x) \in \mathscr{S}_{n,r}$ such that

$$S(\nu) = y_\nu, \quad S'(\nu) = y_\nu',\ldots, S^{(r-1)}(\nu) = y_\nu^{(r-1)}, \quad (\nu = 0, \pm 1, \pm 2,\ldots). \tag{1.4}$$

We shall say that $S(x)$ belongs to the class $\mathscr{S}_{n,r}^g$ of Cardinal g-splines of degree n corresponding to $(0, 1,\ldots, r-2, r)$-interpolation problem when

$$S(x) \in C^{n-r-1}(-\infty, \infty), \tag{1.5}$$

$$S(x) \text{ is equal to a polynomial of degree } n \text{ in each of the intervals } [\nu, \nu+1]; \text{ and} \tag{1.6}$$

$$S^{(n-r+1)}(\nu+) = S^{(n-r+1)}(\nu-), \qquad (\nu = 0, \pm 1, \pm 2,\ldots), \tag{1.7}$$

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where we write $S^{(n-r+1)}(x)$ to mean the $(n-r+1)$th derivative of the polynomial components of $S(x)$.

The $(0, 1,..., r-2, r)$ Cardinal lacunary interpolation problem (C.L.I.P.) in $\mathscr{S}^g_{n,r}$ is as follows: Given r bi-infinite sequences of numbers

$$y = (y_\nu), \qquad y' = (y_\nu'),..., y^{(r-2)} = (y_\nu^{(r-2)}), \qquad y^{(r)} = (y_\nu^{(r)}) \tag{1.8}$$

we wish to find an $S(x) \in \mathscr{S}^g_{n,r}$ such that

$$S(\nu) = y_\nu, \qquad S'(\nu) = y_\nu',..., S^{(r-2)}(\nu) = y_\nu^{(r-2)}, \qquad S^{(r)}(\nu) = y_\nu^{(r)},$$
$$(\nu = 0, \pm 1, \pm 2,...). \tag{1.9}$$

In connection with the C.L.I.P. we define the following null spaces

$$\mathring{\mathscr{S}}^*_{n,r} = \{S(x) \in \mathscr{S}_{n,r}: S^{(\rho)}(\nu) = 0, \quad \text{for all integers } \nu, \quad \text{and} \quad \rho = 0, 1,..., r-2, r\}, \tag{1.10}$$

$$\mathring{\mathscr{S}}^{**}_{n,r} = \{S(x) \in \mathscr{S}^g_{n,r}: S^{(\rho)}(\nu) = 0, \quad \text{for all integers } \nu, \quad \text{and} \quad \rho = 0, 1,..., r-1\}, \tag{1.11}$$

and

$$\mathring{\mathscr{S}}^g_{n,r} = \{S(x) \in \mathscr{S}^g_{n,r}: S^{(\rho)}(\nu) = 0, \quad \text{for all integers } \nu, \quad \text{and} \quad \rho = 0, 1,..., r-2, r\}. \tag{1.12}$$

Following Schoenberg, we call a spline $S(x) \in \mathring{\mathscr{S}}^g_{n,r}$ (resp. $\mathring{\mathscr{S}}^*_{n,r}$, $\mathring{\mathscr{S}}^{**}_{n,r}$), $S(x) \not\equiv 0$, an eigenspline if it satisfies the functional relation

$$S(x+1) = \lambda S(x), \qquad \text{for all } x, \quad \lambda \neq 0. \tag{1.13}$$

The number λ is called the eigenvalue corresponding to the eigenspline $S(x)$.

In Section 2, we shall see that the search for eigensplines in the null spaces (1.10)–(1.12) leads us to the study of the characteristic polynomials $\Pi^*_{n,r}(\lambda)$, $\Pi^{**}_{n,r}(\lambda)$, and $\Pi^g_{n,r}(\lambda)$. The main results are stated in Section 2. Section 3 deals with some lemmas while Sections 4–6 are devoted to the proof of Theorems 1, 2, and 3, respectively. An interesting relation of the polynomial $\Pi_{n,r+1}(\lambda)$ of Lipow and Schoenberg to the Hankel determinant of the Euler–Frobenius polynomial $\Pi_n(\lambda)$ is given in Section 7.

We plan to apply these results to solve the $(0, 1,..., r-2, r)$ C.L.I.P. in $\mathscr{S}^g_{n,r}$ in a subsequent communication.

2. Statement of Results

Let $P = \|\binom{i}{j}\|$, $(i, j = 0, 1, 2, \ldots)$, be an infinite matrix of the binomial coefficients so that the characteristic matrix is $P - \lambda I = \|\binom{i}{j} - \lambda\delta_{ij}\|$, $(i, j = 0, 1, 2, \ldots)$. We shall denote by $P\binom{i_1, i_1, \ldots, i_\nu}{j_1, j_2, \ldots, j_\nu} : \lambda)$ the determinant of the submatrix of $P - \lambda I$ obtained by deleting all the rows and columns except those numbered $i_1, i_2, \ldots, i_\nu$ and $j_1, j_2, \ldots, j_\nu$, respectively, and by $P\binom{i_1, i_2, \ldots, i_\nu}{j_2, j_2, \ldots, j_\nu}$ we denote the corresponding determinant obtained from P.

In connection with C.H.I.P., Lipow and Schoenberg [6] proved that the polynomial

$$\Pi_{n,r}(\lambda) = P\begin{pmatrix} r, r+1, \ldots, n \\ 0, 1, \ldots, n-r \end{pmatrix} : \lambda \Big) \tag{2.1}$$

is a reciprocal polynomial of degree $(n - 2r + 1)$ with real and simple zeros of sign $(-1)^r$. For $r = 1$, $\Pi_{n,1}(\lambda) \equiv \Pi_n(\lambda)$ is the so-called Euler–Frobenius polynomial (see [7, 2]). It was shown by Frobenius [3] that $\Pi_n(\lambda)$ has real simple negative zeros interlacing with the zeros of $\Pi_{n-1}(\lambda)$.

Now, let us determine the eigensplines in $\mathring{\mathscr{S}}^g_{n,r}$ · $\mathring{\mathscr{S}}^g_{n,r}$ is a vector space of dimension $n - 2r + 1$ and each spline function in $\mathring{\mathscr{S}}^g_{n,r}$ is uniquely determined by its polynomial component in [0, 1]. Let $S(x) \in \mathring{\mathscr{S}}^g_{n,r}$, $S(x) \not\equiv 0$, be an eigenspline satisfying (1.13), and suppose that $P(x)$ is its polynomial component in [0, 1]. In view of (1.5), (1.7) and (1.12), we have

$$P^{(\rho)}(1) = P^{(\rho)}(0) = 0, \qquad (\rho = 0, 1, \ldots, r-2, r). \tag{2.2}$$

and

$$P^{(\rho)}(1) = \lambda P^{(\rho)}(0), \qquad (\rho = r-1, r+1, \ldots, n-r-1, n-r+1). \tag{2.3}$$

By (2.2) we can write

$$P(x) = a_0 x^n + a_1 \binom{n}{1} x^{n-1} + \cdots$$
$$+ a_{n-r-1} \binom{n}{n-r-1} x^{r+1} + a_{n-r+1} \binom{n}{n-r+1} x^{r-1}. \tag{2.4}$$

Writing equations $P^{(\rho)}(1) = 0$, $(\rho = 0, 1, \ldots, r-2, r)$, and $P^{(\rho)}(1) - \lambda P^{(\rho)}(0) = 0$, $(\rho = r-1, r+1, \ldots, n-r-1, n-r+1)$, upwards with increasing ρ, we obtain a system of $(n - r + 1)$ linear homogeneous equations in $(n - r + 1)$ unknowns $a_0, a_1, \ldots, a_{n-r-1}, a_{n-r+1}$. The determinant of the matrix of the system is the $(n - 2r + 1)$th degree polynomial in λ given by

$$\Pi^g_{n,r}(\lambda) = P\left(\begin{matrix} r-1, r+1,\dots, n \\ 0, 1,\dots, n-r-1, n-r+1 \end{matrix} : \lambda\right) =$$

$$\begin{vmatrix}
1 & \binom{r-1}{1} & \cdots & \binom{r-1}{r-2} & (1-\lambda) & 0 & 0 & \cdots & 0 & 0 \\
1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-2} & \binom{r+1}{r-1} & \binom{r+1}{r} & (1-\lambda) & \cdots & 0 & 0 \\
\vdots & & & & & & & \ddots & 0 & \vdots \\
1 & \binom{n-r-1}{1} & \cdots & & & & & \cdots & (1-\lambda) & \vdots \\
1 & \binom{n-r}{1} & \cdots & & & & & \cdots & \binom{n-r}{n-r-1} & 0 \\
1 & \binom{n-r+1}{1} & \cdots & & & & & \cdots & \binom{n-r+1}{n-r-1} & (1-\lambda) \\
1 & \binom{n-r+2}{1} & \cdots & & & & & \cdots & \binom{n-r+2}{n-r-1} & \binom{n-r+2}{n-r+1} \\
\vdots & \vdots & & & & & & & \vdots & \vdots \\
1 & \binom{n}{1} & \cdots & & & & & \cdots & \binom{n}{n-r-1} & \binom{n}{n-r+1}
\end{vmatrix} \tag{2.5}$$

Thus, the eigenvalues corresponding to the eigensplines of the null space $\mathring{\mathscr{S}}^g_{n,r}$ are precisely the zeros of the polynomial $\Pi^g_{n,r}(\lambda)$.

A similar argument shows that the eigenvalues corresponding to the eigensplines of the null spaces $\mathring{\mathscr{S}}^*_{n,r}$ and $\mathring{\mathscr{S}}^{**}_{n,r}$ are precisely the zeros of the $(n-2r+1)$th degree polynomials

$$\Pi^*_{n,r}(\lambda) = P\left(\begin{matrix} r, r+1,\dots, n \\ 0, 1,\dots, n-r-1, n-r+1 \end{matrix} : \lambda\right), \tag{2.6}$$

and

$$\Pi^{**}_{n,r}(\lambda) = P\left(\begin{matrix} r-1, r+1,\dots, n \\ 0, 1,\dots, n-r \end{matrix} : \lambda\right), \tag{2.7}$$

respectively.

We shall prove the following theorems.

THEOREM 1. *Let $r \geqslant 1$ be a fixed integer. For $n \geqslant 2r+1$, the zeros of $\Pi_{n,r}(\lambda)$ are real simple and of* sign$(-1)^r$, *and interlace with the zeros of $\Pi_{n-1,r}(\lambda)$.*

THEOREM 2. *Let $r \geqslant 2$ be a fixed integer. For each $n \geqslant 2r+1$, the algebraic equations*

$$\Pi^{*}_{n,r}(\lambda) = 0, \tag{2.8}$$

and

$$\Pi^{**}_{n,r}(\lambda) = 0 \tag{2.9}$$

are reciprocal equations in λ of degree $d = n - 2r + 1$ whose roots are real and simple with $(-1)^{r+1}$ as a common root. The remaining $d-1$ zeros of $\Pi^{*}_{n,r}(\lambda)$ (resp. $\Pi^{**}_{n,r}(\lambda)$) are of sign$(-1)^r$ *and interlace with the zeros of $\Pi_{n,r}(\lambda)$.*

Remark. It is shown later in (7.10) that $\Pi^{*}_{n,r}(\lambda)$ and $\Pi^{**}_{n,r}(\lambda)$ differ only by a constant factor.

THEOREM 3. *Let $r \geqslant 2$ be a fixed integer. For each $n \geqslant 2r+1$, the algebraic equation*

$$\Pi^{g}_{n,r}(\lambda) = 0 \tag{2.10}$$

is a reciprocal equation in λ of degree $d = n - 2r + 1$ with exactly two roots of sign$(-1)^{r+1}$, *and $(d-2)$ roots of* sign$(-1)^r$. *If n is odd, all the zeros of $\Pi_{n,r}(\lambda)$ are simple and are separated by those of $\Pi_{n,r}(\lambda)$ (resp. $\Pi^{**}_{n,r}(\lambda)$). If n is even, $(-1)^{r+1}$ is a double zero and the remaining $d-2$ zeros are distinct and interlace with the $d-1$ zeros of $\Pi^{*}_{n,r}(\lambda)$ (resp. $\Pi^{**}_{n,r}(\lambda)$) of* sign$(-1)^r$.

COROLLARY. *If n is odd, the algebraic equation* (2.10) *has real simple roots not lying on the unit circle.*

Remark. Essentially, Theorem 1 is the theorem of Lipow and Schoenberg [6]. What is new here is the interlacing of the zeros of $\Pi_{n,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$ and we shall give a new proof in Section 4.

3. SOME LEMMAS

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{f}^{(1)},\ldots, \mathbf{f}^{(n)}$ be $(n+4)$ arbitrary column vectors in $\mathbb{R}^{n+2}$. Let $D(\mathbf{a}, \mathbf{b}, \mathbf{f}) \equiv D(\mathbf{a}, \mathbf{b}, \mathbf{f}^{(1)},\ldots, \mathbf{f}^{(n)})$ denote the determinant of the $(n+2) \times (n+2)$ matrix formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{f}^{(1)},\ldots, \mathbf{f}^{(n)}$ arranged in this order.

LEMMA 3.1 (Karlin [4, p. 7]).

$$\begin{vmatrix} D(\mathbf{a}, \mathbf{c}, \mathbf{f}) & D(\mathbf{a}, \mathbf{d}, \mathbf{f}) \\ D(\mathbf{b}, \mathbf{c}, \mathbf{f}) & D(\mathbf{b}, \mathbf{d}, \mathbf{f}) \end{vmatrix} = D(\mathbf{a}, \mathbf{b}, \mathbf{f})\, D(\mathbf{c}, \mathbf{d}, \mathbf{f}). \tag{3.1}$$

The idendity (3.1) will be the main tool in the proof of the following

LEMMA 3.2. *For $n \geqslant 2r + 1$ and $r \geqslant 1$, we have*

$$r\Pi_{n,r+1}(\lambda)\,\Pi_{n-2,r-1}(\lambda) = n[\Pi_{n-1,r}(\lambda)]^2 - (n-r)\,\Pi_{n-2,r}(\lambda)\,\Pi_{n,r}(\lambda). \tag{3.2}$$

For integer n and r, $n \geqslant 2r + 1$, $r \geqslant 2$, we have

$$\Pi_{n,r-1}(\lambda)\,\Pi_{n-1,r}(\lambda) = \Pi^*_{n,r}(\lambda)\,\Pi_{n-1,r-1}(\lambda) - \Pi_{n,r}(\lambda)\,\Pi^*_{n-1,r-1}(\lambda), \tag{3.3}$$

and

$$\Pi_{n,r-1}(\lambda)\,\Pi_{n,r+1}(\lambda) = \Pi^*_{n,r}(\lambda)\,\Pi^{**}_{n,r}(\lambda) - \Pi^{g}_{n,r}(\lambda)\,\Pi_{n,r}(\lambda). \tag{3.4}$$

Proof. To prove (3.2) let us consider the following $(n - r + 1) \times (n - r + 1)$ matrix

$$\left\| \begin{array}{ccccccccc}
1 & \binom{r}{1} & \cdots & \binom{r}{r-1} & (1-\lambda) & 0 & \cdots & 0 & 0 \\
1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & (1-\lambda) & \cdots & 0 & 0 \\
\vdots & & & & & & \ddots & \vdots & \vdots \\
1 & \binom{n-r-1}{1} & \cdots & \binom{n-r-1}{r-1} & \binom{n-r-1}{r} & \binom{n-r-1}{r+1} & \cdots & (1-\lambda) & 0 \\
1 & \binom{n-r}{1} & \cdots & \binom{n-r}{r-1} & \binom{n-r}{r} & \binom{n-r}{r+1} & \cdots & \binom{n-r}{n-r-1} & (1-\lambda) \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & \binom{n}{1} & \cdots & \binom{n}{r-1} & \binom{n}{r} & \binom{n}{r+1} & \cdots & \binom{n}{n-r-1} & \binom{n}{n-r}
\end{array} \right\| \tag{3.5}$$

Let us denote the first and the last columns of (3.5), respectively, by **c** and **d** and its νth column by $f^{(\nu-1)}$ for $\nu = 2, 3,\ldots, n - r$. Set $\mathbf{a} = (1, 0,\ldots, 0)$ and $\mathbf{b} = (0,\ldots, 0, 1)$. Then, after some simplifications, we have $D(\mathbf{a}, \mathbf{b}, \mathbf{f}) = (-1)^{n-r-1}\binom{n-1}{r}\Pi_{n-2,r}(\lambda)$, $D(\mathbf{a}, \mathbf{c}, \mathbf{f}) = \Pi_{n,r+1}(\lambda)$, $D(\mathbf{a}, \mathbf{d}, \mathbf{f}) = (-1)^{n-r-1}\binom{n}{r} \times \Pi_{n-1,r}(\lambda)$, $D(\mathbf{b}, \mathbf{c}, \mathbf{f}) = (-1)^{n-r}\Pi_{n-1,r}(\lambda)$, $D(\mathbf{b}, \mathbf{d}, \mathbf{f}) = -\binom{n-1}{r-1}\Pi_{n-2,r-1}(\lambda)$, $D(\mathbf{c}, \mathbf{d}, \mathbf{f}) = (-1)^{n-r-1}\Pi_{n,r}(\lambda)$. Substituting these into (3.1) we obtain (3.2).

To prove (3.3) we again set $a = (1, 0,\ldots, 0)^T$, $\mathbf{b} = (0, 0,\ldots, 0, 1)^T$ and let $\mathbf{f}^{(\nu)}$ denote the νth column of (3.5) for $\nu = 1, 2,\ldots, n - r - 1$, with **c**, **d** representing the last two columns. Then $D(\mathbf{a}, \mathbf{b}, \mathbf{f}) = (-1)^{n-r-1}\Pi_{n-1,r+1}(\lambda)$, $D(\mathbf{a}, \mathbf{c}, \mathbf{f}) = (-1)^{n-r-1}\Pi_{n,r+1}(\lambda)$, $D(\mathbf{a}, \mathbf{d}, \mathbf{f}) = (-1)^{n-r-1}\Pi^*_{n,r+1}(\lambda)$, $D(\mathbf{b}, \mathbf{c}, \mathbf{f}) = -\Pi_{n-1,r}(\lambda)$, $D(\mathbf{b}, \mathbf{d}, \mathbf{f}) = -\Pi^*_{n-1,r}(\lambda)$, and $D(\mathbf{c}, \mathbf{d}, \mathbf{f}) = \Pi_{n,r}(\lambda)$.

Replacing r by $(r - 1)$ in the above determinants and using the identity (3.1) we obtain (3.3).

To prove (3.4) we use the same vectors as in the proof of (3.3) except that $\mathbf{b}$ is replaced by the vector $(0, 1, 0,..., 0)^T$. Then (3.4) is obtained by replacing r by $r - 1$.

Remark. Let $P_n = \|\binom{i}{j}\|$, $(i, j = 0, 1,..., n)$. It is easy to show (see, for example, [6]) that P_n is totally positive. Also, it is clear that if λ is an eigenvalue corresponding to an eigenspline $S(x) \in \mathscr{S}^g_{n,r}$ (resp. $\mathscr{S}^*_{n,r}$, $\mathscr{S}^{**}_{n,r}$), then $1/\lambda$ is an eigenvalue corresponding to the eigenspline $\mathscr{S}(x) = S(-x) \in \mathscr{S}^g_{n,r}$ (resp. $\mathscr{S}^*_{n,r}$, $\mathscr{S}^{**}_{n,r}$). It follows that $\Pi^g_{n,r}(\lambda)$, $\Pi^*_{n,r}(\lambda)$, $\Pi^{**}_{n,r}(\lambda)$ are reciprocal polynomials and so their values at $\lambda = 0$ are positive.

LEMMA 3.3. *For $n > 2r$*

$$\Pi_{n,r}(\lambda) > 0 \qquad \textit{for} \quad (-1)^{r+1}\lambda \geqslant 0. \tag{3.6}$$

Proof. A straightforward computation shows that

$$\Pi_{n,r}(\lambda) = a_0[(-1)^{r+1}\lambda]^{n-2r+1} + a_1[(-1)^{r+1}\lambda]^{n-2r} + \cdots + a_{n-2r+1}, \tag{3.7}$$

where

$$a_k = \sum P\begin{pmatrix} r+\nu_1, r+\nu_2,..., r+\nu_k, n-r+1,..., n \\ 0, 1,..., r-1, r+\nu_1, r+\nu_2,..., r+\nu_k \end{pmatrix}, \tag{3.8}$$

for $k = 1, 2,..., n - 2r + 1$, the summation being taken over all the $\binom{n-2r+1}{k}$ choices of $\{\nu_1, \nu_2,..., \nu_k\}$ from $\{0, 1,..., n - 2r\}$, and

$$a_0 = P\begin{pmatrix} n-r+1,..., n \\ 0, 1,..., r-1 \end{pmatrix}. \tag{3.9}$$

Since $\Pi_{n,r}(\lambda)$ is a reciprocal polynomial, $\Pi_{n,r}(0) \neq 0$ and so $a_{n-2r+1} > 0$, in view of the fact that P_n is totally positive. Eq. (3.6) follows from (3.7).

4. PROOF OF THEOREM 1

In this and the following sections, we denote the zeros of $\Pi_{n,r}(\lambda)$ by $\{\lambda_i^{(n)}\}$.

The proof of Theorem 1 follows by induction on n. We shall only give the proof for the case when r is even. The case when r is odd can be treated in the same way.

When $n = 2r$, in view of the fact that $P\binom{k, k+1,..., n}{0, 1,..., n-k} = 1$ for all k and n, we easily see that $\Pi_{2r,r}(\lambda) = (1 - \lambda)$. If we set $n = 2r + 1$ in (3.2) of Lemma 3.2 and take into account that $\Pi_{2r+1,r+1}(\lambda) \equiv \Pi_{2r-1,r}(\lambda) \equiv 1$, we get

$$r\Pi_{2r-1,r-1}(\lambda) = (2r + 1)[\Pi_{2r,r}(\lambda)]^2 - (r + 1)\Pi_{2r+1,r}(\lambda), \tag{4.1}$$

whence

$$r\Pi_{2r-1,r-1}(1) = -(r+1)\,\Pi_{2r+1,r}(1), \tag{4.2}$$

and since $\Pi_{2r-1,r-1}(1) > 0$, it follows that $\Pi_{2r+1,r}(1) < 0$. But then $\Pi_{2r+1,r}(0) > 0$, and therefore, $\Pi_{2r+1,r}(\lambda)$ has a zero in (0, 1) and since it is a reciprocal polynomial it has a zero in $(1, \infty)$. Thus, the zeros of $\Pi_{2r+1,r}(\lambda)$ are real, simple, and positive and they interlace with the zero of $\Pi_{2r,r}(\lambda)$.

Let us suppose that the assertion has been proved for $\Pi_{n-2,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$. More precisely, let us suppose that the zeros of $\Pi_{n-2,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$ are such that

$$0 < \lambda_1^{(n-1)} < \lambda_1^{(n-2)} < \lambda_2^{(n-1)} < \cdots < \lambda_{n-2r-1}^{(n-2)} < \lambda_{n-2r}^{(n-1)}. \tag{4.3}$$

It follows from (3.2) that

$$r\Pi_{n,r+1}(\lambda_i^{(n-1)})\,\Pi_{n-2,r-1}(\lambda_i^{(n-1)}) = -(n-r)\,\Pi_{n-2,r}(\lambda_i^{(n-1)})\,\Pi_{n,r}(\lambda_i^{(n-1)}),$$
$$(r = 1, 2,\ldots, n-2r). \tag{4.4}$$

Since r is even, it follows from Lemma 3.3 that L.H.S. of (4.4) is positive for $i = 1, 2,\ldots, n-2r$. Hence,

$$\Pi_{n-2,r}(\lambda_i^{(n-1)})\,\Pi_{n,r}(\lambda_i^{(n-1)}) < 0, \qquad (i = 1, 2,\ldots, n-2r). \tag{4.5}$$

Since, by inductive hypothesis, $\Pi_{n-2,r}(\lambda_i^{(n-1)})$ has alternating sign for $i = 1, 2,\ldots, n-2r$, it follows that $\Pi_{n,r}(\lambda_i^{(n-1)})$ has alternating sign for $i = 1, 2,\ldots, n-2r$. Hence, $\Pi_{n,r}(\lambda)$ has at least one zero in each of the intervals $(\lambda_i^{(n-1)}, \lambda_{i+1}^{(n-1)})$, $(i = 1, 2,\ldots, n-2r-1)$. Also, by hypothesis and the fact that $\Pi_{n-2,r}(0) > 0$, we have $\Pi_{n-2,r}(\lambda_1^{(n-1)}) > 0$. It follows that $\Pi_{n,r}(\lambda_1^{(n-1)}) < 0$. But then $\Pi_{n,r}(0) > 0$. Hence, $\Pi_{n,r}(\lambda)$ has a zero in $(0, \lambda_1^{(n-1)})$ and since it is a reciprocal polynomial, it also must have a zero in $(\lambda_{n-2r}^{(n-1)}, \infty)$. The assertion is proved by induction.

5. Proof of Theorem 2

To prove the result for Eq. (2.8) we again assume that r is even and suppose that the zeros of $\Pi_{n-1,r}(\lambda)$ and $\Pi_{n,r}(\lambda)$ are given by

$$0 < \lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_{n-2r}^{(n-1)} < \lambda_{n-2r+1}^{(n)}. \tag{5.1}$$

Certainly we can assume this by Theorem 1. Now (3.3) of Lemma 3.2 implies that

$$\Pi_{n,r-1}(\lambda_i^{(n)})\,\Pi_{n-1,r}(\lambda_i^{(n)}) = \Pi_{n\ r}^{*}(\lambda_i^{(n)})\,\Pi_{n-1,r-1}(\lambda_i^{(n)}),$$
$$(i = 1, 2,\ldots, n-2r+1). \tag{5.2}$$

Since $\Pi_{n,r-1}(\lambda_i^{(n)})$ and $\Pi_{n-1,r-1}(\lambda_i^{(n)})$ are positive for all $i = 1,..., n - 2r + 1$, it follows ftom (5.2) that

$$\operatorname{Sgn} \Pi_{n-1,r}(\lambda_i^{(n)}) = \operatorname{Sgn} \Pi_{n,r}^{*}(\lambda_i^{(n)}). \tag{5.3}$$

By the same argument as in the proof of Theorem 1, $\Pi_{n,r}^{*}(\lambda)$ has exactly one zero in each of the intervals $(\lambda_i^{(n)}, \lambda_{i+1}^{(n)})$, $(i = 1, 2,..., n - 2r)$. It cannot have a zero in $(0, \lambda_1^{(n)})$ or $(\lambda_{n-2r+1}^{(n)}, \infty)$. For if it has a zero in $(0, \lambda_1^{(n)})$, it must also have a zero in $(\lambda_{n-2r+1}^{(n)}, \infty)$ and vice versa, which is impossible since it is only of degree $n - 2r + 1$. Since $\Pi_{n,r}^{*}(\lambda)$ is a reciprocal equation, the remaining zero must be -1. Thus, we have established theorem 2 for Eq. (2.8).

Using the result, which we have just established, and (3.4) instead of (3.3), and applying the same reasoning as above, we will arrive at the assertion for the polynomial equation (2.9).

6. Proof of Theorem 3

Again, we assume that r is even.

(i) *n is odd.* Let us denote by $\{\mu_i^{(n)}\}_{i=1}^{n-2r+1}$, with $\mu_1^{(n)} = -1$, the zeros of $\Pi_{n,r}^{*}(\lambda)$ (or of $\Pi_{n,r}^{**}(\lambda)$). By Theorem 2, we can write

$$\mu_1^{(n)} < \lambda_1^{(n)} < \mu_2^{(n)} < \cdots < \mu_{n-2r+1}^{(n)} < \lambda_{n-2r+1}^{(n)}. \tag{6.1}$$

From (3.4) of Lemma 3.2, we have

$$\Pi_{n,r}^{g}(\mu_i^{(n)})\, \Pi_{n,r}(\mu_i^{(n)}) = -\Pi_{n,r-1}(\mu_i^{(n)})\, \Pi_{n,r+1}(\mu_i^{(n)}),$$
$$(i = 1, 2,..., n - 2r + 1). \tag{6.2}$$

Since $\mu_i^{(n)} > 0$ for $i = 2, 3,..., n - 2r + 1$, it follows, using the usual argument that $\Pi_{n,r}^{g}(\lambda)$ has at least one zero in each of the intervals $(\mu_i^{(n)}, \mu_{i+1}^{(n)})$, $(i = 2, 3,..., n - 2r)$. Thus, $\Pi_{n,r}^{g}(\lambda)$ has at least $(n - 2r - 1)$ distinct positive zeros.

We shall show that $\Pi_{n,r}^{g}(\lambda)$ possesses two distinct negative zeros. First, let us observe that since $\Pi_{n,r+1}(0)$ and $\Pi_{n,r-1}(0)$ are both positive, it follows that $\operatorname{Sgn} \Pi_{n,r+1}(-1) = (-1)^{(n-2r-1)/2}$, and $\operatorname{Sgn} \Pi_{n,r-1}(-1) = (-1)^{(n-2r+3)/2}$. Hence, from (6.2) and the fact that $\Pi_{n,r}(-1)$ is positive, it follows that $\Pi_{n,r}^{g}(-1)$ is negative. But then $\Pi_{n,r}^{g}(0)$ is positive and therefore, $\Pi_{n,r}^{g}(\lambda)$ must have a zero in $(-1, 0)$. Since it is a reciprocal polynomial it must also have a zero in $(-\infty, -1)$. Thus, we have shown that $\Pi_{n,r}^{g}(\lambda)$ has at least two distinct negative zeros and at least $(n - 2r - 1)$ distinct positive zeros.

Since its degree is $(n - 2r + 1)$ we conclude that it has exactly two distinct negative and $(n - 2r - 1)$ distinct positive zeros which interlace with the zeros of $\Pi^*_{n,r}(\lambda)$ (or $\Pi^{**}_{n,r}(\lambda)$).

(ii) *n is even.* In this case $\Pi_{n,r-1}(-1) = \Pi_{n,r+1}(-1) = \Pi^*_{n,r}(-1) = \Pi^{**}_{n,r}(-1) = 0$. Since $\Pi_{n,r}(-1)$ is positive, it follows from (3.4) that -1 is a zero of $\Pi^g_{n,r}(\lambda)$. By the same argument as in (i) above we see that $\Pi^g_{n,r}(\lambda)$ has exactly $(n - 2r - 1)$ distinct positive zeros that interlace with the $(n - 2r)$ positive zeros of $\Pi^*_{n,r}(\lambda)$ (or $\Pi^{**}_{n,r}(\lambda)$). Since $\Pi^g_{n,r}(\lambda)$ is a reciprocal polynomial, the remaining zero must be (-1). Thus, (-1) is a zero of multiplicity 2. Hence, Theorem 3 is proved.

7. Hankel Determinant Involving Euler–Frobenius Polynomials

Let us denote by $H_{r+1}(a_n)$ the Hankel determinant of order $(r + 1)$ and let $H^*_{r+1}(a_n)$ and $H^g_{r+1}(a_n)$ be two determinants of the same order associated with $H_{r+1}(a_n)$. More precisely, we set

$$H_{r+1}(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r-1} \\ \vdots & \vdots & & \vdots \\ a_{n-r} & a_{n-r-1} & \cdots & a_{n-2r} \end{vmatrix}, \tag{7.1}$$

$$H^*_{r+1}(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} & a_{n-r-1} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} & a_{n-r-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-r} & a_{n-r-1} & \cdots & a_{n-2r+1} & a_{n-2r-1} \end{vmatrix} \tag{7.2}$$

and

$$H^g_{r+1}(a^u) = \begin{vmatrix} a_n & \cdots & a_{n-r+1} & a_{n-r-1} \\ \vdots & & \vdots & \vdots \\ a_{n-r+1} & \cdots & a_{n-2r+2} & a_{n-2r} \\ a_{n-r-1} & \cdots & a_{n-2r} & a_{n-2r-2} \end{vmatrix} \tag{7.3}$$

Observe that a gap occurs in the last column of the determinant in (7.2) while similar gaps occur in the last row and last column of the determinant in (7.3).

Determinants of type (7.1) and (7.2) in which the entries a_n are orthogonal polynomials have been studied by several mathematicians (see [1], [5]). In this section, we shall consider the determinants in which $a_n = \Pi_n(\lambda)/n!$ where $\Pi_n(\lambda)$ is the Euler–Frobenius polynomial.

We have

THEOREM 4. *Let n and r be positive integers such that $n \geqslant 2r+1$. Then*

$$H_{r+1}(\Pi_n(\lambda)/n!) = (-1)^{[(r+1)/2]} C(n, r)(1-\lambda)^{r(n-r)} \Pi_{n,r+1}(\lambda), \tag{7.4}$$

where

$$C(n, r) = \frac{1!\,2! \cdots r!}{n!\,(n-1)! \cdots (n-r)!}. \tag{7.5}$$

The proof of Theorem 4 depends on the following lemma, which is a particular case of a more general identity on symmetric determinants (see [8, p. 372].

LEMMA 7.1. *For $n \geqslant 2r+1$ we have*

$$H_r(a_n)\, H_r(a_{n-2}) - [H_r(a_{n-1})]^2 = H_{r+1}(a_{n-2}) H_{r+1}(a_n). \tag{7.6}$$

Proof of Lemma. The lemma is easily established using the determinantal identity (3.1) applied to the following vectors: **a** and **d** are the first and last columns of (7.1), respectively, $\mathbf{f}^{(\nu)}$ is the $(\nu+1)$th column $(\nu = 1, 2,..., r-1)$, and $\mathbf{a} = (1, 0,..., 0)^{\mathrm{T}}$, $\mathbf{b} = (0, 0,..., 0, 1)^{\mathrm{T}}$.

Proof of Theorem 4. The proof will be carried out by induction on r. First, observe that $H_1(\Pi_n(\lambda)/n!) = \Pi_n(\lambda)/n!$ $(n \geqslant 1)$ and by (3.2) of Lemma 3.2, with $r = 1$, we have

$$H_2\left(\frac{\Pi_n(\lambda)}{n!}\right) = \frac{(-1)(1-\lambda)^{n-1}\, \Pi_{n,2}(\lambda)}{n!\,(n-1)!} \qquad (n \geqslant 3).$$

Let us suppose that (7.4) holds for $H_k(\Pi_n(\lambda)/n!)$ $(k = 1, 2,..., r)$ and we shall prove that (7.4) also holds for $H_{r+1}(\Pi_n(\lambda)/n!)$. Using (7.6), after some calculations, we have, for $n \geqslant 2r+1$,

$$\begin{aligned} &H_{r+1}\left(\frac{\Pi_n(\lambda)}{n!}\right) \Pi_{n-2,r-1}(\lambda) \qquad (7.7)\\ &= (-1)^{[(r+1)/2]} \frac{C(n, r)(r-1)!}{r!} \{n(\Pi_{n-1,r}(\lambda))^2 - (n-r)\,\Pi_{n,r}(\lambda)\, \Pi_{n-2,r}(\lambda))\}, \end{aligned}$$

where $C(n, r)$ is given by (7.5). From this we obtain (7.4) and (7.5) with the help of (3.2).

By the same method we obtain the following identities:

$$\begin{aligned} H_{r+1}^*(\Pi_n(\lambda)/n!) &= (-1)^{[(r+1)/2]} C(n, r)(n-r)(1-\lambda)^{r(n-r)-1} \Pi_{n,r+1}^*(\lambda) \\ &= (-1)^{[(r+1)/2]} C(n, r)(r+1)(1-\lambda)^{r(n-r)-1} \Pi_{n,r+1}^{**}(\lambda) \\ &\quad (n \geqslant 2r+2), \end{aligned} \tag{7.8}$$

$$H_r^g(\Pi_n(\lambda)/n!) = (-1)^{[(r+1)/2]}\, C(n, r)(r + 1)(n - r)(1 - \lambda)^{r(n-r)-2}\, \Pi_{n,r+1}^g(\lambda)$$
$$(n \geqslant 2r + 3). \tag{7.9}$$

From (7.8) it follows that

$$(n - r)\, \Pi_{n,r+1}^*(\lambda) = (r + 1)\, \Pi_{n,r+1}^{**}(\lambda). \tag{7.10}$$

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